

Solutionbank M4

Edexcel AS and A Level Modular Mathematics

Elastic collisions in two dimensions

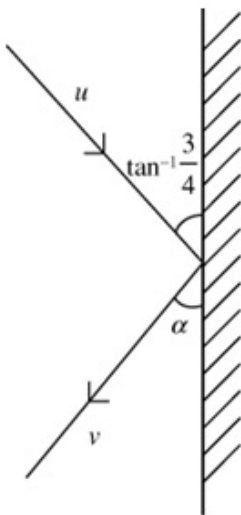
Exercise A, Question 1

Question:

A smooth sphere S is moving on a smooth horizontal plane with speed u when it collides with a smooth fixed vertical wall. At the instant of collision the direction of motion of S makes an angle of $\tan^{-1} \frac{3}{4}$ with the wall. The coefficient of restitution between S and the wall is $\frac{1}{3}$.

Find the speed of S immediately after the collision.

Solution:



$$R \uparrow: v \cos \alpha = u \times \frac{4}{5}$$

$$\text{law of restitution } \leftrightarrow v \sin \alpha = e \times u \times \frac{3}{5} = \frac{1}{3} \times u \times \frac{3}{5} = u \times \frac{1}{5}$$

squaring and adding,

$$v^2 = u^2 \left(\frac{16}{25} + \frac{1}{25} \right) = u^2 \times \frac{17}{25}$$

$$v = \frac{u\sqrt{17}}{5}$$

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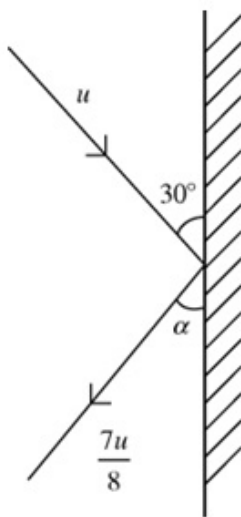
Exercise A, Question 2

Question:

A smooth sphere S is moving on a smooth horizontal plane with speed u when it collides with a smooth fixed vertical wall. At the instant of collision the direction of motion of S makes an angle of 30° with the wall. Immediately after the collision the speed of S is $\frac{7}{8}u$.

Find the coefficient of restitution between S and the wall.

Solution:



$$R \uparrow: \frac{7u}{8} \cos \alpha = u \cos 30^\circ$$

$$\text{law of restitution } \leftrightarrow: \frac{7u}{8} \sin \alpha = eu \sin 30^\circ$$

squaring and adding:

$$\frac{49u^2}{64} = u^2 \left(\frac{3}{4} + \frac{e^2}{4} \right)$$

$$\frac{49}{16} = 3 + e^2$$

$$\frac{1}{16} = e^2, e = \frac{1}{4}$$

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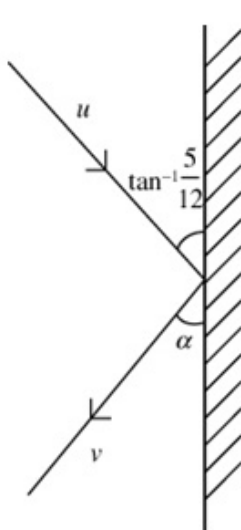
Exercise A, Question 3

Question:

A smooth sphere S is moving on a smooth horizontal plane with speed u when it collides with a smooth fixed vertical wall. At the instant of collision the direction of motion of S makes an angle of $\tan^{-1} \frac{5}{12}$ with the wall. The coefficient of restitution between S and the wall is $\frac{3}{5}$.

Find the speed of S immediately after the collision.

Solution:



$$R \uparrow: v \cos \alpha = u \times \frac{12}{13}$$

$$\text{law of restitution } \Leftrightarrow v \sin \alpha = e \times u \times \frac{5}{13} = \frac{3}{5} \times u \times \frac{5}{13} = u \times \frac{3}{13}$$

squaring and adding,

$$v^2 = u^2 \left(\frac{144}{169} + \frac{9}{169} \right) = u^2 \times \frac{153}{169}$$

$$v = \frac{3\sqrt{17}u}{13}$$

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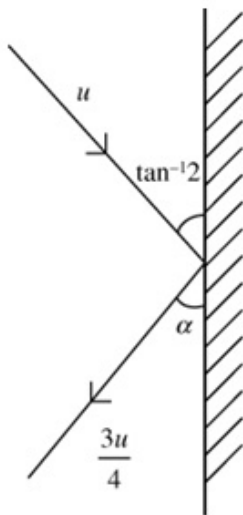
Exercise A, Question 4

Question:

A smooth sphere S is moving on a smooth horizontal plane with speed u when it collides with a smooth fixed vertical wall. At the instant of collision the direction of motion of S makes an angle of $\tan^{-1} 2$ with the wall. Immediately after the collision the speed of S is $\frac{3}{4}u$.

Find the coefficient of restitution between S and the wall.

Solution:



$$R \uparrow: \frac{3u}{4} \cos \alpha = u \times \frac{1}{\sqrt{5}}$$

$$\text{law of restitution } \leftrightarrow \frac{3u}{4} \sin \alpha = eu \times \frac{2}{\sqrt{5}}$$

squaring and adding:

$$\frac{9u^2}{16} = u^2 \left(\frac{1}{5} + \frac{4e^2}{5} \right)$$

$$\frac{45}{16} = 1 + 4e^2$$

$$\frac{29}{16} = 4e^2, e = \frac{\sqrt{29}}{8}$$

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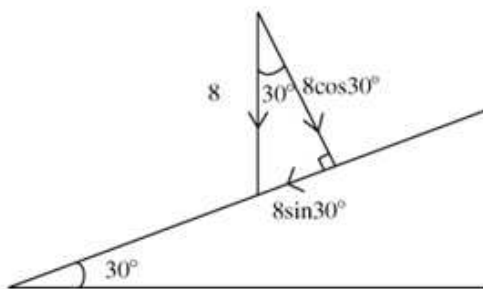
Exercise A, Question 5

Question:

A small smooth ball is falling vertically. The ball strikes a smooth plane which is inclined at an angle 30° to the horizontal. Immediately before striking the plane the ball has speed 8 m s^{-1} . The coefficient of restitution between the ball and the plane is $\frac{1}{4}$. Find the exact value of the speed of the ball immediately after the impact.

Solution:

Before the impact



The component of velocity parallel to the

$$\text{slope} = 8 \sin 30^\circ = 8 \times \frac{1}{2} = 4$$

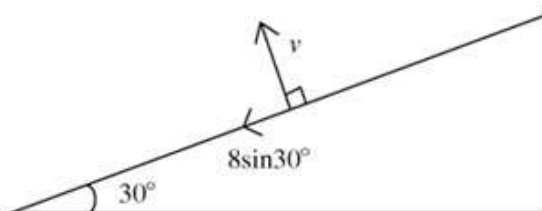
Perpendicular to the slope:

$$v = e \times 8 \cos 30^\circ = \frac{1}{4} \times 8 \times \frac{\sqrt{3}}{2} = \sqrt{3}$$

Therefore the speed immediately after

$$\text{impact} = \sqrt{4^2 + \sqrt{3}^2} = \sqrt{19} \text{ m s}^{-1}$$

After the impact



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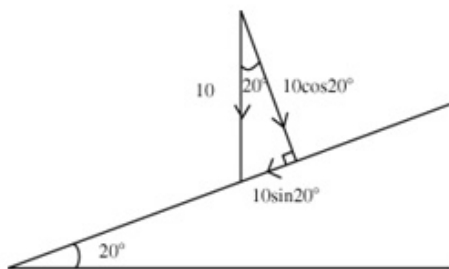
Exercise A, Question 6

Question:

A small smooth ball is falling vertically. The ball strikes a smooth plane which is inclined at an angle 20° to the horizontal. Immediately before striking the plane the ball has speed 10 m s^{-1} . The coefficient of restitution between the ball and the plane is $\frac{2}{5}$. Find the speed, to 3 significant figures, of the ball immediately after the impact.

Solution:

Before the impact



The component of velocity parallel to the slope $= 10 \sin 20^\circ$

Perpendicular to the slope:

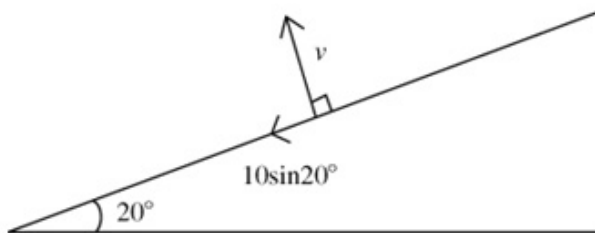
$$v = e \times 10 \cos 20^\circ = \frac{2}{5} \times 10 \cos 20^\circ = 4 \cos 20^\circ$$

Therefore the speed immediately after impact

$$= \sqrt{(10 \sin 20^\circ)^2 + (4 \cos 20^\circ)^2}$$

$$= \sqrt{25.826\dots} = 5.08 \text{ m s}^{-1}$$

After the impact



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Exercise A, Question 7

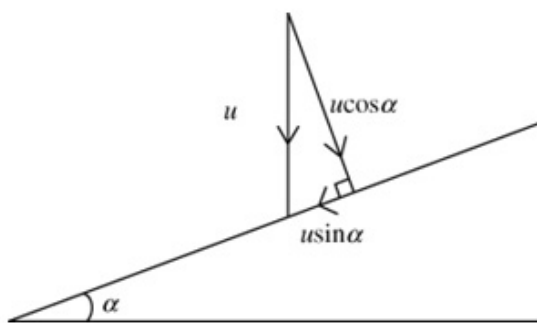
Question:

A small smooth ball of mass 750 g is falling vertically. The ball strikes a smooth plane which is inclined at an angle 45° to the horizontal. Immediately before striking the plane the ball has speed $5\sqrt{2} \text{ m s}^{-1}$. The coefficient of restitution between the ball and the plane is $\frac{1}{2}$. Find

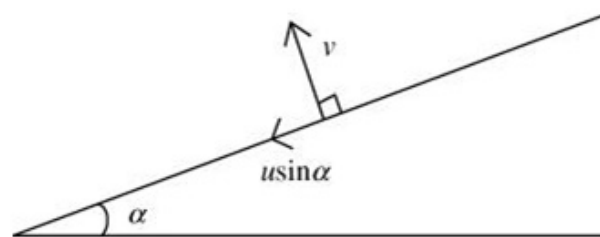
- the speed, to 3 significant figures, of the ball immediately after the impact,
- the magnitude of the impulse received by the ball as it strikes the plane.

Solution:

Before the impact



After the impact



- The component of velocity parallel to the slope = $u \sin \alpha = 5\sqrt{2} \sin 45^\circ = 5\sqrt{2} \times \frac{1}{\sqrt{2}} = 5$

Perpendicular to the slope:

$$v = e \times u \cos \alpha = \frac{1}{2} \times 5\sqrt{2} \cos 45^\circ$$

$$= \frac{1}{2} \times 5\sqrt{2} \times \frac{1}{\sqrt{2}} = \frac{5}{2}$$

Therefore the speed immediately after

$$\text{impact} = \sqrt{5^2 + 2.5^2} = \sqrt{31.25} = 5.59 \text{ m s}^{-1}$$

- The impulse is perpendicular to the surface:

$$I = \frac{3}{4} \left(\frac{5}{2} - (-5) \right) = \frac{3}{4} \times \frac{15}{2} = 5.625 \text{ Ns}$$

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Elastic collisions in two dimensions

Exercise A, Question 8

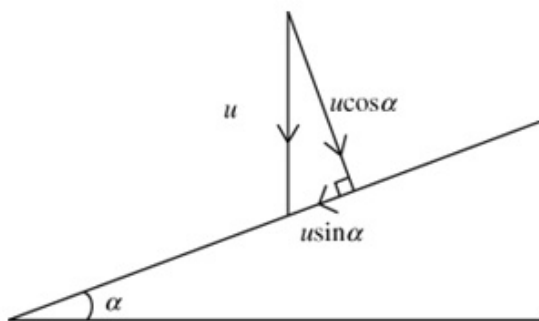
Question:

A small smooth ball is falling vertically. The ball strikes a smooth plane which is inclined at an angle α to the horizontal, where $\tan \alpha = \frac{3}{4}$. Immediately before striking the plane the ball has speed 7.5 m s^{-1} . Immediately after the impact the ball has speed 5 m s^{-1} .

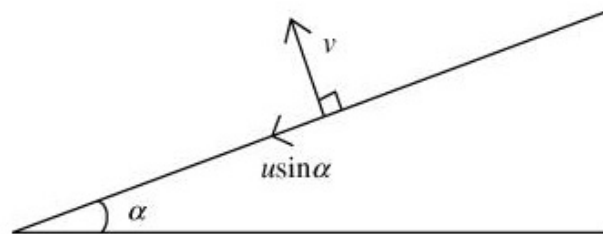
Find the coefficient of restitution to 2 significant figures, between the ball and the plane.

Solution:

Before the impact



After the impact



$$\therefore e^2 = \frac{25 - 20.25}{36} = 0.1319\dots$$

$$e = 0.36$$

The component of velocity parallel to the

$$\text{slope} = u \sin \alpha = 7.5 \times \frac{3}{5} = 4.5$$

Perpendicular to the slope:

$$v = eu \cos \alpha = e \times 7.5 \times \frac{4}{5} = 6e$$

Combining the two components:

$$5^2 = 4.5^2 + 36e^2$$

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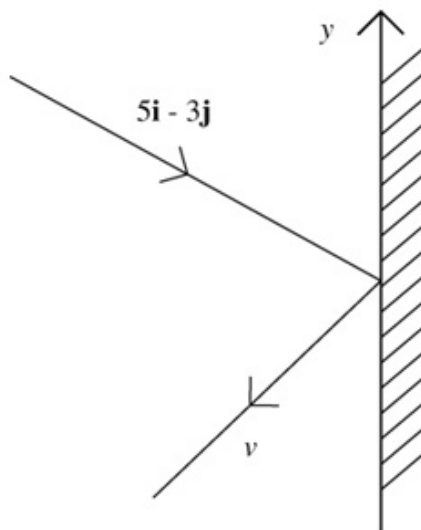
Exercise A, Question 9

Question:

A small smooth ball of mass 800 g is moving in the xy -plane and collides with a smooth fixed vertical wall which contains the y -axis. The velocity of the ball just before impact is $(5\mathbf{i} - 3\mathbf{j})\text{m s}^{-1}$. The coefficient of restitution between the sphere and the wall is $\frac{1}{2}$. Find

- the velocity of the ball immediately after the impact,
- the kinetic energy lost as a result of the impact.

Solution:



- a** Suppose that the velocity of the ball immediately after the impact is $p\mathbf{i} + q\mathbf{j}$

$$\uparrow -3 = q$$

$$\leftrightarrow -p = e \times 5 = \frac{5}{2}$$

$$\text{so } v = -2.5\mathbf{i} - 3\mathbf{j}$$

- b** K.E. before = $\frac{1}{2} \times \frac{4}{5} \times (5^2 + 3^2) = 13.6$

$$\text{K.E. after} = \frac{1}{2} \times \frac{4}{5} \times (2.5^2 + 3^2) = 6.1$$

$$\text{K.E. lost} = 13.6 - 6.1 = 7.5 \text{ J}$$

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Elastic collisions in two dimensions

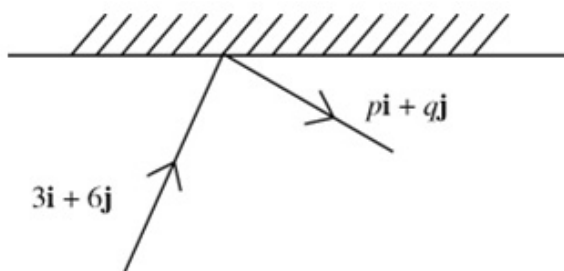
Exercise A, Question 10

Question:

A small smooth ball of mass 1 kg is moving in the xy -plane and collides with a smooth fixed vertical wall which contains the x -axis. The velocity of the ball just before impact is $(3\mathbf{i} + 6\mathbf{j})\text{ m s}^{-1}$. The coefficient of restitution between the sphere and the wall is $\frac{1}{3}$. Find

- the speed of the ball immediately after the impact,
- the kinetic energy lost as a result of the impact.

Solution:



- Suppose that the velocity of the ball immediately after the impact is $p\mathbf{i} + q\mathbf{j}$
 $\leftrightarrow 3 = p$ (parallel to the wall)
 $\uparrow -q = \frac{1}{3} \times 6 = 2$ (perpendicular to the wall)

$$\text{Speed} = \sqrt{3^2 + 2^2} = \sqrt{13} \text{ m s}^{-1}.$$

- K.E. before impact $= \frac{1}{2} \times 1 \times (3^2 + 6^2) = 22.5$, K.E. after impact $= \frac{1}{2} \times 1 \times 13 = 6.5$
 K.E. lost $= 22.5 - 6.5 = 16 \text{ J}$

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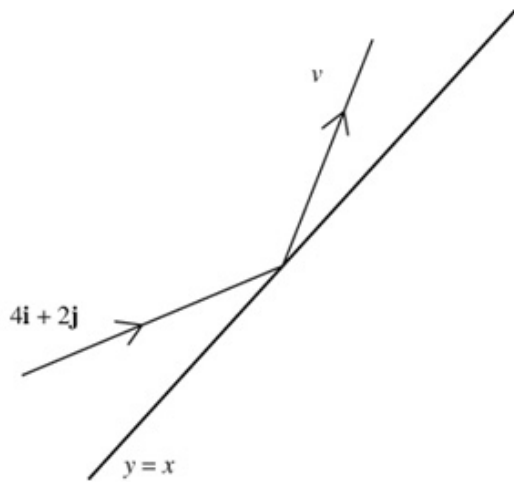
Exercise A, Question 11

Question:

A small smooth ball of mass 2 kg is moving in the xy -plane and collides with a smooth fixed vertical wall which contains the line $y = x$. The velocity of the ball just before impact is $(4\mathbf{i} + 2\mathbf{j})\text{m s}^{-1}$. The coefficient of restitution between the sphere and the wall is $\frac{1}{3}$. Find

- the velocity of the ball immediately after the impact,
- the kinetic energy lost as a result of the impact.

Solution:



Suppose that $\mathbf{v} = \mathbf{a} + \mathbf{b}$
where \mathbf{a} is parallel to the wall and \mathbf{b}
is perpendicular to the wall.

$\frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j})$ is a unit vector in
the direction of the wall.

$\frac{1}{\sqrt{2}}(-\mathbf{i} + \mathbf{j})$ is a unit vector
perpendicular to the wall.

$$\begin{aligned} \nearrow \left[(4\mathbf{i} + 2\mathbf{j}) \cdot \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) \right] \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) &= \mathbf{a} \\ &= \frac{1}{\sqrt{2}} \times 6 \times \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) = 3\mathbf{i} + 3\mathbf{j} \\ \nwarrow \frac{1}{3} \left[(4\mathbf{i} + 2\mathbf{j}) \cdot \frac{1}{\sqrt{2}}(-\mathbf{i} + \mathbf{j}) \right] \frac{1}{\sqrt{2}}(-\mathbf{i} + \mathbf{j}) &= \mathbf{b} \\ &= \frac{1}{3} \times \frac{1}{\sqrt{2}} \times (4 - 2) \times \frac{1}{\sqrt{2}}(-\mathbf{i} + \mathbf{j}) \\ &= \frac{1}{3}(-\mathbf{i} + \mathbf{j}) \end{aligned}$$

$$\text{So } \mathbf{v} = 3\mathbf{i} + 3\mathbf{j} - \frac{1}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} = \frac{8}{3}\mathbf{i} + \frac{10}{3}\mathbf{j}$$

$$\begin{aligned} \text{b K.E. before} &= \frac{1}{2} \times 2 \times (4^2 + 2^2) = 20, \text{ K.E. after} = \frac{1}{2} \times 2 \times \left(\frac{64}{9} + \frac{100}{9} \right) = \frac{164}{9} \\ \text{K.E. lost} &= 20 - \frac{164}{9} = \frac{16}{9} \text{ J} \end{aligned}$$

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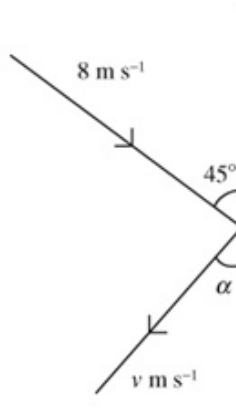
Elastic collisions in two dimensions

Exercise A, Question 12

Question:

A smooth snooker ball strikes a smooth cushion with speed 8 m s^{-1} at an angle of 45° to the cushion. Given that the coefficient of restitution between the ball and the cushion is $\frac{2}{5}$, find the magnitude and direction of the velocity of the ball after the impact.

Solution:



\uparrow : no change parallel to the cushion
 $8 \cos 45^\circ = v \cos \alpha = 4\sqrt{2}$
 \leftrightarrow : Using the law of restitution
 $\frac{2}{5} \times 8 \sin 45^\circ = v \sin \alpha = \frac{8\sqrt{2}}{5}$
 Squaring and adding:
 $v^2 = (4\sqrt{2})^2 + \left(\frac{8\sqrt{2}}{5}\right)^2 = 37.12, v = 6.09$
 Dividing: $\tan \alpha = \frac{\frac{8\sqrt{2}}{5}}{4\sqrt{2}} = \frac{2}{5}, \alpha = \tan^{-1} 0.4$
 Velocity is 6.09 m s^{-1} at 21.8° to the cushion.

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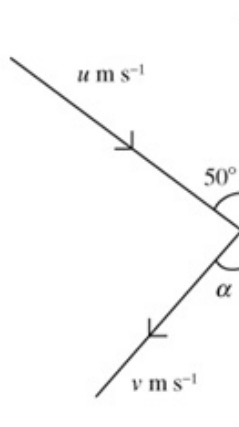
Exercise A, Question 13

Question:

A smooth snooker ball strikes a smooth cushion with speed $u \text{ m s}^{-1}$ at an angle of 50° to the cushion. The coefficient of restitution between the ball and the cushion is e .

- Show that the angle between the cushion and the rebound direction is independent of u .
- Find the value of e given that the ball rebounds at right angles to its original direction.

Solution:



- a** \uparrow : no change parallel to the cushion

$$u \cos 50^\circ = v \cos \alpha$$

$$\leftrightarrow: \text{Using the law of restitution, } e \times u \sin 50^\circ = v \sin \alpha$$

$$\text{Dividing: } \frac{v \sin \alpha}{v \cos \alpha} = \frac{eu \sin 50^\circ}{u \cos 50^\circ}$$

$$\Rightarrow \tan \alpha = e \tan 50^\circ, \text{ which is independent of the value of } u$$

- b** If $\alpha = 40^\circ$ then $\tan 40^\circ = e \tan 50^\circ$

$$e = \frac{\tan 40^\circ}{\tan 50^\circ} \approx 0.7$$

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Exercise A, Question 14

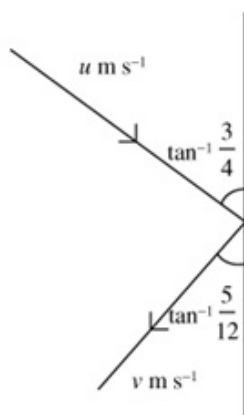
Question:

A smooth billiard ball strikes a smooth cushion at an angle of $\tan^{-1} \frac{3}{4}$ to the cushion.

The ball rebounds at an angle of $\tan^{-1} \frac{5}{12}$ to the cushion. Find

- the fraction of the kinetic energy of the ball lost in the collision,
- the coefficient of restitution between the ball and the cushion.

Solution:



a \uparrow : no change parallel to the cushion

$$u \times \frac{4}{5} = v \times \frac{12}{13}$$

$$v = u \times \frac{4}{5} \times \frac{13}{12} = \frac{13}{15}u$$

$$\text{K.E. lost} = \frac{1}{2} \times m \times u^2 - \frac{1}{2} \times m \times \frac{169}{225} u^2 = \frac{1}{2} \times m \times \frac{56}{225} u^2 \text{ J}$$

$$\text{Fraction of K.E. lost} = \frac{\frac{1}{2} \times m \times \frac{56}{225} u^2}{\frac{1}{2} \times m \times u^2} = \frac{56}{225}$$

b \leftrightarrow : Using the law of restitution, $e \times u \times \frac{3}{5} = v \times \frac{5}{13}$

$$e \times u \times \frac{3}{5} = \frac{13}{15} u \times \frac{5}{13}$$

$$e = \frac{5}{15} \times \frac{5}{3} = \frac{5}{9}$$

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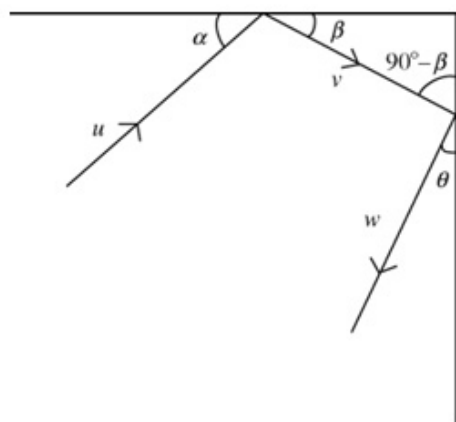
Exercise A, Question 15

Question:

Two vertical walls meet at right angles at the corner of a room. A small smooth disc slides across the floor and bounces off each wall in turn. Just before the first impact the disc is moving with speed $u \text{ m s}^{-1}$ at an acute angle α to the wall. The coefficient of restitution between the disc and the wall is e . Find

- the direction of the motion of the disc after the second collision,
- the speed of the disc after the second collision.

Solution:



a First collision:

$$\uparrow: e \times u \sin \alpha = v \sin \beta$$

$$\leftrightarrow: u \cos \alpha = v \cos \beta$$

Second collision:

$$\uparrow: v \cos(90 - \beta) = v \sin \beta = w \cos \theta$$

$$\leftrightarrow: e \times v \sin(90 - \beta) = ev \cos \beta = w \sin \theta$$

$$\Rightarrow \tan \theta = \frac{e \cos \beta}{\sin \beta} = \frac{e}{\tan \beta}$$

$$= \frac{e}{e \tan \alpha} = \frac{1}{\tan \alpha}$$

$$\Rightarrow \theta = 90^\circ - \alpha, \text{ so the path is parallel to the original path but in the opposite direction}$$

- $eu \sin \alpha = v \sin \beta = w \cos \theta = w \cos(90^\circ - \alpha) = w \sin \alpha$
speed after second collision = $w = eu$

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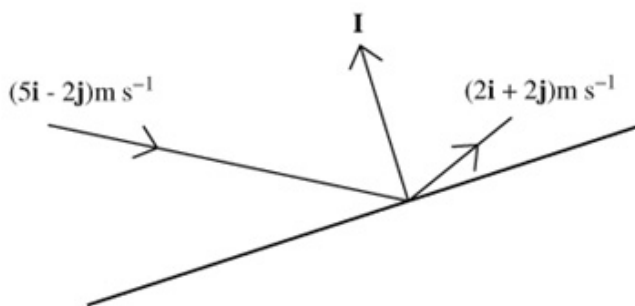
Exercise A, Question 16

Question:

A small smooth sphere of mass m is moving with velocity $(5\mathbf{i} - 2\mathbf{j})\text{m s}^{-1}$ when it hits a smooth wall. It rebounds from the wall with velocity $(2\mathbf{i} + 2\mathbf{j})\text{m s}^{-1}$. Find

- the magnitude and direction of the impulse received by the sphere,
- the coefficient of restitution between the sphere and the wall.

Solution:



$$\begin{aligned} \mathbf{I} &= m\mathbf{v} - m\mathbf{u} = m\{(2\mathbf{i} + 2\mathbf{j}) - (5\mathbf{i} - 2\mathbf{j})\} \\ &= m(-3\mathbf{i} + 4\mathbf{j}) \end{aligned}$$

The impulse has magnitude $5m$ Ns in the direction parallel to the unit vector

$$\frac{1}{5}(-3\mathbf{i} + 4\mathbf{j}).$$

- Component of $(5\mathbf{i} - 2\mathbf{j})$ in the direction of the impulse

$$\begin{aligned} &= [(5\mathbf{i} - 2\mathbf{j}) \cdot \frac{1}{5}(-3\mathbf{i} + 4\mathbf{j})] \frac{1}{5}(-3\mathbf{i} + 4\mathbf{j}) = \frac{1}{5} \times (-15 - 8) \frac{1}{5}(-3\mathbf{i} + 4\mathbf{j}) \\ &= \frac{-23}{5} \times \frac{1}{5}(-3\mathbf{i} + 4\mathbf{j}) \end{aligned}$$

Component of $(2\mathbf{i} + 2\mathbf{j})$ in the direction of the impulse

$$\begin{aligned} &= [(2\mathbf{i} + 2\mathbf{j}) \cdot \frac{1}{5}(-3\mathbf{i} + 4\mathbf{j})] \frac{1}{5}(-3\mathbf{i} + 4\mathbf{j}) = \frac{1}{5} \times (-6 + 8) \frac{1}{5}(-3\mathbf{i} + 4\mathbf{j}) \\ &= \frac{2}{5} \times \frac{1}{5}(-3\mathbf{i} + 4\mathbf{j}) \end{aligned}$$

law of restitution

$$\Rightarrow \frac{2}{5} = e \times \frac{23}{5}$$

$$e = \frac{2}{23}$$

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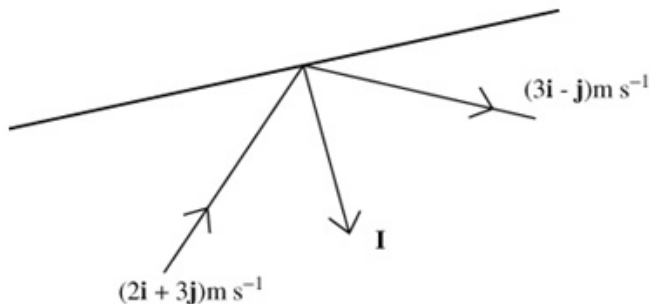
Exercise A, Question 17

Question:

A small smooth sphere of mass 2 kg is moving with velocity $(2\mathbf{i} + 3\mathbf{j})\text{m s}^{-1}$ when it hits a smooth wall. It rebounds from the wall with velocity $(3\mathbf{i} - \mathbf{j})\text{m s}^{-1}$. Find

- the magnitude and direction of the impulse received by the sphere,
- the coefficient of restitution between the sphere and the wall,
- the kinetic energy lost by the sphere in the collision.

Solution:



$$\begin{aligned} \mathbf{I} &= m\mathbf{v} - m\mathbf{u} = m\{(3\mathbf{i} - \mathbf{j}) - (2\mathbf{i} + 3\mathbf{j})\} \\ &= 2(\mathbf{i} - 4\mathbf{j}) \end{aligned}$$

The impulse has magnitude

$$2\sqrt{17} \text{ Ns in}$$

the direction parallel to the unit vector

$$\frac{1}{\sqrt{17}}(\mathbf{i} - 4\mathbf{j})$$

$$\mathbf{b} \text{ Component of } (2\mathbf{i} + 3\mathbf{j}) \text{ in the direction of the impulse} =$$

$$[(2\mathbf{i} + 3\mathbf{j}) \cdot \frac{1}{\sqrt{17}}(\mathbf{i} - 4\mathbf{j})] \frac{1}{\sqrt{17}}(\mathbf{i} - 4\mathbf{j}) = \frac{1}{\sqrt{17}}(2 - 12) \frac{1}{\sqrt{17}}(\mathbf{i} - 4\mathbf{j}) = \frac{-10}{\sqrt{17}} \times \frac{1}{\sqrt{17}}(\mathbf{i} - 4\mathbf{j})$$

Component of $(3\mathbf{i} - \mathbf{j})$ in the direction of the impulse =

$$[(3\mathbf{i} - \mathbf{j}) \cdot \frac{1}{\sqrt{17}}(\mathbf{i} - 4\mathbf{j})] \frac{1}{\sqrt{17}}(\mathbf{i} - 4\mathbf{j}) = \frac{1}{\sqrt{17}}(3 + 4) \frac{1}{\sqrt{17}}(\mathbf{i} - 4\mathbf{j}) = \frac{7}{\sqrt{17}} \times \frac{1}{\sqrt{17}}(\mathbf{i} - 4\mathbf{j})$$

$$\text{law of restitution} \Rightarrow \frac{7}{\sqrt{17}} = e \times \frac{10}{\sqrt{17}}, e = \frac{7}{10}$$

$$\mathbf{c} \text{ K.E. just before the impact} = \frac{1}{2} \times 2 \times (2^2 + 3^2) = 13$$

$$\text{K.E. just after the impact} = \frac{1}{2} \times 2 \times (3^2 + 1^2) = 10$$

$$\text{K.E. lost} = 13 - 10 = 3 \text{ J}$$

Solutionbank M4

Edexcel AS and A Level Modular Mathematics

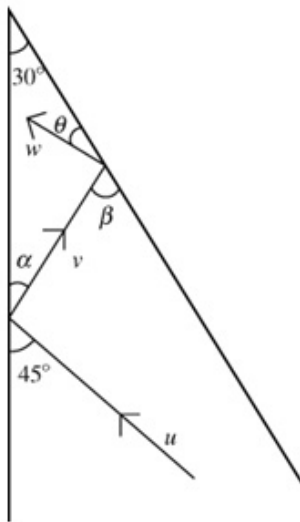
Elastic collisions in two dimensions

Exercise A, Question 18

Question:

Two smooth vertical walls stand on a smooth horizontal floor and intersect at an angle of 30° . A particle is projected along the floor with speed $u \text{ m s}^{-1}$ at 45° to one of the walls and towards the intersection of the walls. The coefficient of restitution between the particle and each wall is $\frac{1}{\sqrt{3}}$. Find the speed of the particle after one impact with each wall.

Solution:



For the first impact:

$$u \cos 45^\circ = \frac{u}{\sqrt{2}} = v \cos \alpha$$

$$eu \sin 45^\circ = \frac{1}{\sqrt{3}} \times \frac{u}{\sqrt{2}} = v \sin \alpha$$

By dividing, $\tan \alpha = \frac{1}{\sqrt{3}}$, $\alpha = 30^\circ$, and $\beta = 60^\circ$

$$\text{Squaring and adding, } v^2 = \frac{u^2}{2} + \frac{u^2}{6} = \frac{4u^2}{6}$$

For the second impact:

$$v \cos 60^\circ = \frac{v}{2} = w \cos \theta$$

$$ev \sin 60^\circ = v \times \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{2} = \frac{v}{2} = w \sin \theta$$

Squaring and adding,

$$w^2 = \left(\frac{v}{2}\right)^2 + \left(\frac{v}{2}\right)^2 = \frac{v^2}{2} = \frac{u^2}{3}, \quad w = \frac{\sqrt{3}u}{3} \text{ m s}^{-1}$$

Solutionbank M4

Edexcel AS and A Level Modular Mathematics

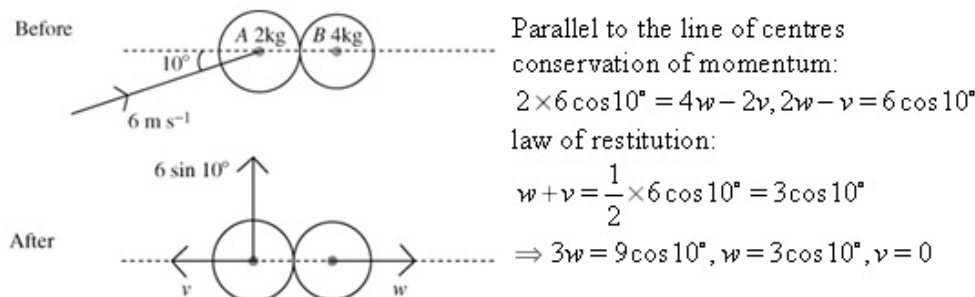
Elastic collisions in two dimensions

Exercise B, Question 1

Question:

A smooth sphere A , of mass 2 kg and moving with speed 6 m s^{-1} collides obliquely with a smooth sphere B of mass 4 kg . Just before the impact B is stationary and the velocity of A makes an angle of 10° with the lines of centres of the two spheres. The coefficient of restitution between the spheres is $\frac{1}{2}$. Find the magnitudes and directions of the velocities of A and B immediately after the impact.

Solution:



So, after the impact, A has velocity $6 \sin 10^\circ \approx 1.04\text{ m s}^{-1}$ perpendicular to the line of centres, and B has velocity $3 \cos 10^\circ \approx 2.95\text{ m s}^{-1}$ parallel to the line of centres

Solutionbank M4

Edexcel AS and A Level Modular Mathematics

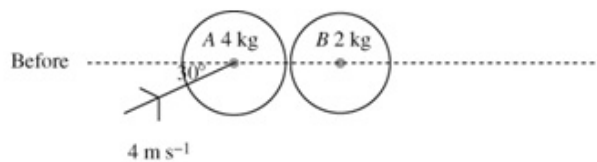
Elastic collisions in two dimensions

Exercise B, Question 2

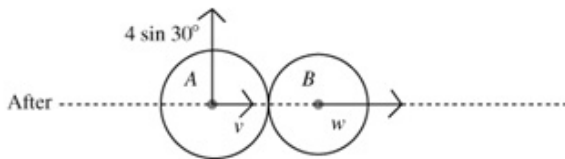
Question:

A smooth sphere A , of mass 4 kg and moving with speed 4 m s^{-1} collides obliquely with a smooth sphere B of mass 2 kg . Just before the impact B is stationary and the velocity of A makes an angle of 30° with the lines of centres of the two spheres. The coefficient of restitution between the spheres is $\frac{1}{3}$. Find the magnitudes and directions of the velocities of A and B immediately after the impact.

Solution:



Perpendicular to the line of centres, component of the velocity of A is

$$4 \sin 30^\circ = 2 \text{ m s}^{-1}$$


Parallel to the line of centres:

$$4 \times 4 \cos 30^\circ = 4v + 2w, 4\sqrt{3} = 2v + w$$

$$w - v = \frac{1}{3} \times 4 \cos 30^\circ, 2w - 2v = \frac{4\sqrt{3}}{3}$$

$$\Rightarrow 3w = \frac{16}{3}\sqrt{3}, w = \frac{16\sqrt{3}}{9}$$

$$\text{and } v = \frac{10\sqrt{3}}{9}$$

B has speed $\frac{16\sqrt{3}}{9} \text{ m s}^{-1}$ along the line of centres.

A has speed $\sqrt{(2)^2 + \left(\frac{10\sqrt{3}}{9}\right)^2} = \sqrt{4 + \frac{100}{27}} = \sqrt{\frac{208}{27}} = \frac{4\sqrt{13}}{3\sqrt{3}} = \frac{4\sqrt{39}}{9} \text{ m s}^{-1}$ at an angle

of $\tan^{-1}\left(\frac{2}{\frac{10\sqrt{3}}{9}}\right) = \tan^{-1}\left(\frac{18}{10\sqrt{3}}\right) = \tan^{-1}\left(\frac{3\sqrt{3}}{5}\right) = 46.1^\circ$ to the line of centres

Solutionbank M4

Edexcel AS and A Level Modular Mathematics

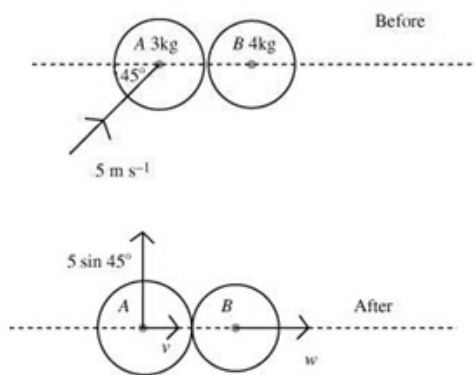
Elastic collisions in two dimensions

Exercise B, Question 3

Question:

A smooth sphere A , of mass 3 kg and moving with speed 5 m s^{-1} collides obliquely with a smooth sphere B of mass 4 kg . Just before the impact B is stationary and the velocity of A makes an angle of 45° with the lines of centres of the two spheres. The coefficient of restitution between the spheres is $\frac{1}{2}$. Find the magnitudes and directions of the velocities of A and B immediately after the impact.

Solution:



Perpendicular to the line of centres, component of the velocity of A is

$$5 \sin 45^\circ = \frac{5\sqrt{2}}{2} \text{ m s}^{-1}$$

Parallel to the line of centres:

$$3 \times 5 \cos 45^\circ = 3v + 4w$$

$$w - v = \frac{1}{2} \times 5 \cos 45^\circ, 3w - 3v = \frac{15\sqrt{2}}{4}$$

$$\Rightarrow 7w = \frac{45\sqrt{2}}{4}, w = \frac{45\sqrt{2}}{28}$$

$$\text{and } v = \frac{10\sqrt{2}}{28} = \frac{5\sqrt{2}}{14}$$

B has speed $\frac{45\sqrt{2}}{28} \text{ m s}^{-1}$ along the line of centres

$$A \text{ has speed } \sqrt{\left(\frac{5\sqrt{2}}{2}\right)^2 + \left(\frac{5\sqrt{2}}{14}\right)^2} = \frac{5\sqrt{2}}{2} \sqrt{1^2 + \left(\frac{1}{7}\right)^2} = \frac{5\sqrt{2}}{2} \sqrt{\frac{50}{49}} = \frac{50}{14} = \frac{25}{7} \text{ m s}^{-1} \text{ at an}$$

$$\text{angle of } \tan^{-1}\left(\frac{5 \sin 45^\circ}{v}\right) = \tan^{-1}\left(\frac{\frac{5\sqrt{2}}{2}}{\frac{5\sqrt{2}}{14}}\right) = \tan^{-1} 7 \approx 81.9^\circ \text{ to the line of centres}$$

Solutionbank M4

Edexcel AS and A Level Modular Mathematics

Elastic collisions in two dimensions

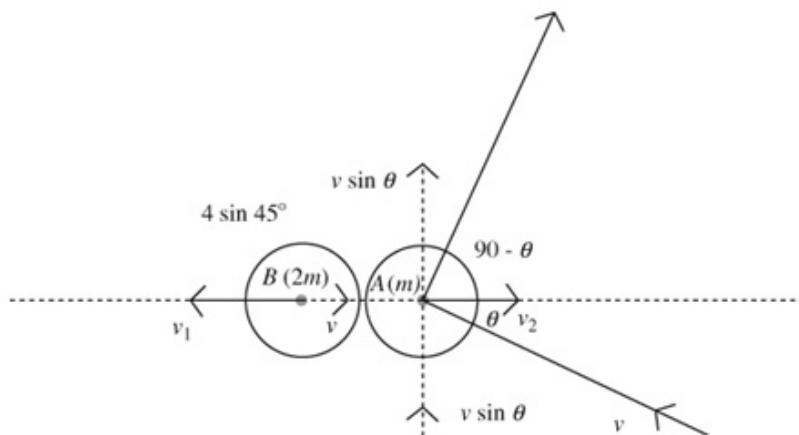
Exercise B, Question 4

Question:

A small smooth sphere A of mass m and a small smooth sphere B of the same radius but mass $2m$ collide. At the instant of impact, B is stationary and the velocity of A makes an angle θ with the line of centres. The direction of motion of A is turned through 90° by the impact. The coefficient of restitution between the spheres is e . Show that

$$\tan^2 \theta = \frac{2e-1}{3}.$$

Solution:



Components perpendicular to the line of centres are unchanged. For A , the component perpendicular to the line of centres is $v \sin \theta$.

Parallel to the line of centres:

$$\text{conservation of momentum} \Rightarrow mv \cos \theta = 2mv_1 - mv_2$$

law of restitution

$$\Rightarrow v_1 + v_2 = ev \cos \theta$$

$$\Rightarrow v \cos \theta = 2(ev \cos \theta - v_2) - v_2 = 2ev \cos \theta - 3v_2$$

$$v_2 = \frac{v \cos \theta (2e-1)}{3}$$

$$\Rightarrow \tan(90 - \theta) = \frac{1}{\tan \theta} = \frac{v \sin \theta}{v_2} = \frac{3v \sin \theta}{v \cos \theta (2e-1)}, \therefore \frac{1}{\tan \theta} = \frac{3 \tan \theta}{2e-1} \therefore \tan^2 \theta = \frac{2e-1}{3}$$

Solutionbank M4

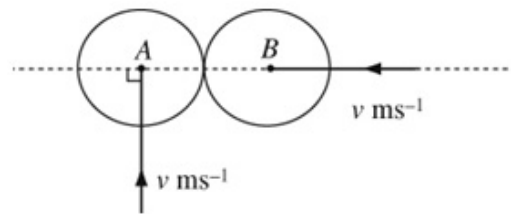
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Elastic collisions in two dimensions

Exercise B, Question 5

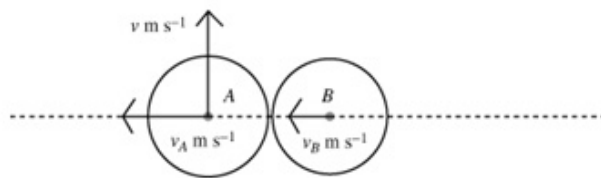
Question:

Two smooth spheres A and B are identical and are moving with equal speeds on a smooth horizontal surface. In the instant before impact, A is moving in a direction perpendicular to the line of centres of the spheres, and B is moving along the line of centres, as



shown in the diagram. The coefficient of restitution between the spheres is $\frac{2}{3}$. Find the speeds and directions of motion of the spheres after the collision.

Solution:



Components perpendicular to the line of centres are unchanged.

Conservation of momentum: $mv = mv_A + mv_B$, $v = v_A + v_B$

law of restitution:

$$v_A - v_B = \frac{2}{3}v$$

$$\Rightarrow 2v_A = \frac{5}{3}v, v_A = \frac{5}{6}v, v_B = \frac{1}{6}v$$

A has speed $\sqrt{1^2 + \left(\frac{5}{6}\right)^2}v = \sqrt{\frac{61}{36}}v = \frac{\sqrt{61}}{6}v$ m s⁻¹ and is moving at

$$\tan^{-1}\left(\frac{1}{\left(\frac{5}{6}\right)}\right) = \tan^{-1}\frac{6}{5} = 50.2^\circ \text{ to the line of centres.}$$

B is moving along the line of centres with speed $\frac{1}{6}v$ m s⁻¹.

Solutionbank M4

Edexcel AS and A Level Modular Mathematics

Elastic collisions in two dimensions

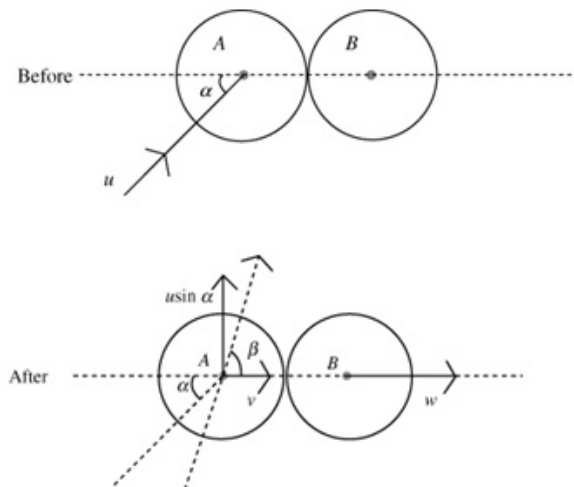
Exercise B, Question 6

Question:

A smooth sphere A collides obliquely with an identical smooth sphere B . Just before the impact B is stationary and the velocity of A makes an angle of α with the lines of centres of the two spheres. The coefficient of restitution between the spheres is e ($e \neq 1$). Immediately after the collision the velocity of A makes an angle of β with the line of centres.

- a Show that $\tan \beta = \frac{2 \tan \alpha}{1 - e}$.
- b Hence show that in the collision the direction of motion of A turns through an angle equal to $\tan^{-1} \left(\frac{(1+e) \tan \alpha}{2 \tan^2 \alpha + 1 - e} \right)$.

Solution:



Perpendicular to the line of centres, component of velocity of A is $u \sin \alpha$. Parallel to the line of centres:

conservation of momentum: $mu \cos \alpha = mv + mw$, $u \cos \alpha = v + w$

law of restitution: $w - v = eu \cos \alpha$, so $2v = u \cos \alpha - eu \cos \alpha = u \cos \alpha(1 - e)$

$$\Rightarrow \tan \beta = \frac{u \sin \alpha}{v} = \frac{2u \sin \alpha}{u \cos \alpha(1 - e)} = \frac{2 \tan \alpha}{1 - e}$$

- b The path of A has been deflected through an angle equal to $\beta - \alpha$.

$$\begin{aligned} \tan(\beta - \alpha) &= \frac{\tan \beta - \tan \alpha}{1 + \tan \alpha \tan \beta} = \frac{\frac{2 \tan \alpha}{1 - e} - \tan \alpha}{1 + \tan \alpha \frac{2 \tan \alpha}{1 - e}} = \frac{2 \tan \alpha - (1 - e) \tan \alpha}{1 - e + 2 \tan^2 \alpha} \\ &= \frac{(1 + e) \tan \alpha}{2 \tan^2 \alpha + 1 - e} \end{aligned}$$

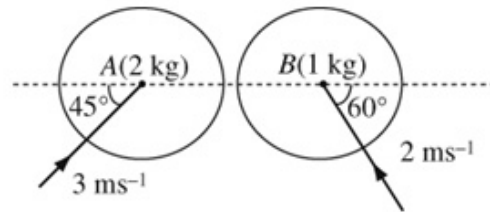
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Elastic collisions in two dimensions

Exercise B, Question 7

Question:

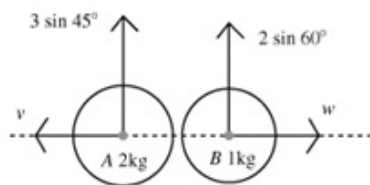


A small smooth sphere A of mass 2 kg collides with a small smooth sphere B of mass 1 kg . Just before the impact A is moving with a speed of 3 m s^{-1} in a direction at 45° to the line of centres and B is moving with speed 2 m s^{-1} at 60° to the line of centres, as shown in the diagram. The coefficient of restitution between the spheres is $\frac{\sqrt{2}}{3}$.

Find

- the kinetic energy lost in the impact,
- the magnitude of the impulse exerted by A on B .

Solution:



No change in the components of velocity perpendicular to the line of centres. Parallel to the line of centres:

$$\text{conservation of momentum: } 1 \times 2 \cos 60^\circ - 2 \times 3 \cos 45^\circ = 2v - w = 1 - 3\sqrt{2}$$

$$\begin{aligned} \text{law of restitution: } v + w &= \frac{\sqrt{2}}{3} (3 \cos 45^\circ + 2 \cos 60^\circ) \\ &= \frac{\sqrt{2}}{3} \left(\frac{3\sqrt{2}}{2} + 1 \right) = 1 + \frac{\sqrt{2}}{3} \end{aligned}$$

Solving the simultaneous equations gives $3v = 2 - \frac{8\sqrt{2}}{3}$, $v = \frac{2}{3} - \frac{8\sqrt{2}}{9} \approx -0.590$ and

$$w = 1 + \frac{\sqrt{2}}{3} - \frac{2}{3} + \frac{8\sqrt{2}}{9} = \frac{1}{3} + \frac{11\sqrt{2}}{9} \approx 2.06$$

a K.E. lost in the impact

$$= \frac{1}{2} \times 2 \times ((3 \cos 45^\circ)^2 - 0.590^2) + \frac{1}{2} \times 1 \times ((2 \cos 60^\circ)^2 - 2.06^2) \approx 2.53\text{ J}$$

b Impulse on $B = 1(w + 2 \cos 60^\circ) \approx 3.06\text{ N s}$

Solutionbank M4

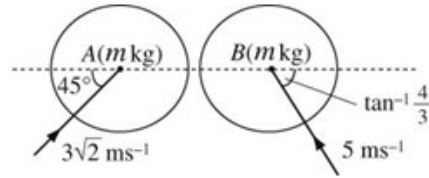
Edexcel AS and A Level Modular Mathematics

Elastic collisions in two dimensions

Exercise B, Question 8

Question:

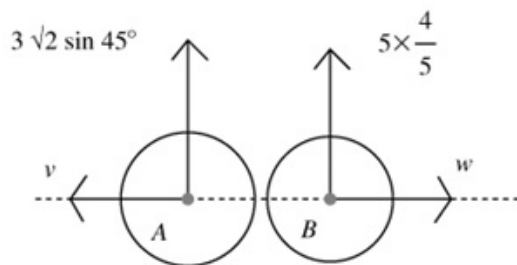
A small smooth sphere A collides with an identical small smooth sphere B . Just before the impact A is moving with a speed of $3\sqrt{2} \text{ m s}^{-1}$ in a direction at 45° to the line of centres and B is moving with speed 5 m s^{-1} at $\tan^{-1} \frac{4}{3}$ to the line of centres, as shown in the diagram.



The coefficient of restitution between the spheres is $\frac{2}{3}$. Find

- the speeds of both spheres immediately after the impact,
- the fraction of the kinetic energy lost in the impact.

Solution:



After the collision the components of velocity perpendicular to the line of centres are 3 m s^{-1} and 4 m s^{-1} . (No change in this direction.)

Parallel to the line of centres:

$$\text{conservation of momentum: } m \times 3\sqrt{2} \cos 45^\circ - m \times 5 \times \frac{3}{5} = mw - mv = 0$$

$$\text{law of restitution: } v + w = \frac{2}{3} \left(3\sqrt{2} \cos 45^\circ + 5 \times \frac{3}{5} \right) = 4$$

$$\text{so } v = w = 2$$

$$\text{a speed of } A = \sqrt{2^2 + 3^2} = \sqrt{13} \text{ m s}^{-1}$$

$$\text{speed of } B = \sqrt{2^2 + 4^2} = \sqrt{20} = 2\sqrt{5} \text{ m s}^{-1}$$

$$\text{b Total K.E. just before impact} = \frac{1}{2} \times m \times (3\sqrt{2})^2 + \frac{1}{2} \times m \times 5^2 = \frac{m \times 43}{2} \text{ J}$$

$$\text{Total K.E. just after impact} = \frac{1}{2} \times m \times (\sqrt{13})^2 + \frac{1}{2} \times m \times (2\sqrt{5})^2 = \frac{m \times 33}{2} \text{ J}$$

$$\text{Fraction of K.E. lost} = \frac{43 - 33}{43} = \frac{10}{43}$$

Solutionbank M4

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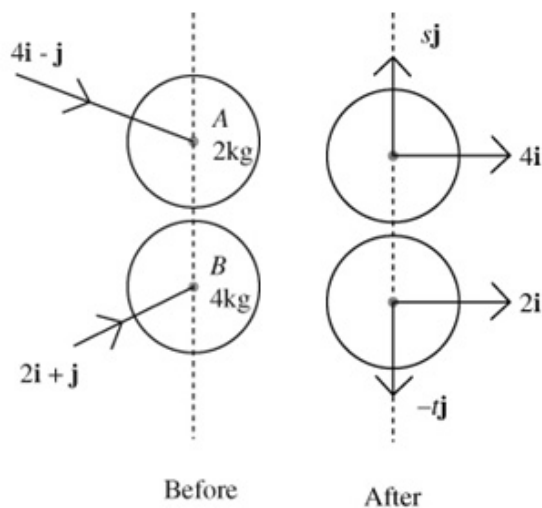
Elastic collisions in two dimensions

Exercise B, Question 9

Question:

A smooth sphere A of mass 2 kg is moving on a smooth horizontal surface with velocity $(4\mathbf{i} - \mathbf{j})\text{ m s}^{-1}$. Another smooth sphere B of mass 4 kg and the same radius as A is moving on the same surface with velocity $(2\mathbf{i} + \mathbf{j})\text{ m s}^{-1}$. The spheres collide when their line of centres is parallel to \mathbf{j} . The coefficient of restitution between the spheres is $\frac{1}{2}$. Find the velocities of both spheres after the impact.

Solution:



Line of centres parallel to $\mathbf{j} \Rightarrow$ no change in the components of velocity parallel to \mathbf{i} .

Conservation of momentum: $-2 \times 1 + 4 \times 1 = 2 \times s - 4 \times t = 2$

law of restitution: $s + t = \frac{1}{2}(1 + 1), s + t = 1$

$$s - 2t = 1$$

$$3s = 3$$

$$s = 1, t = 0$$

velocity of A is $4\mathbf{i} + \mathbf{j}\text{ m s}^{-1}$

velocity of B is $2\mathbf{i}\text{ m s}^{-1}$

Solutionbank M4

Edexcel AS and A Level Modular Mathematics

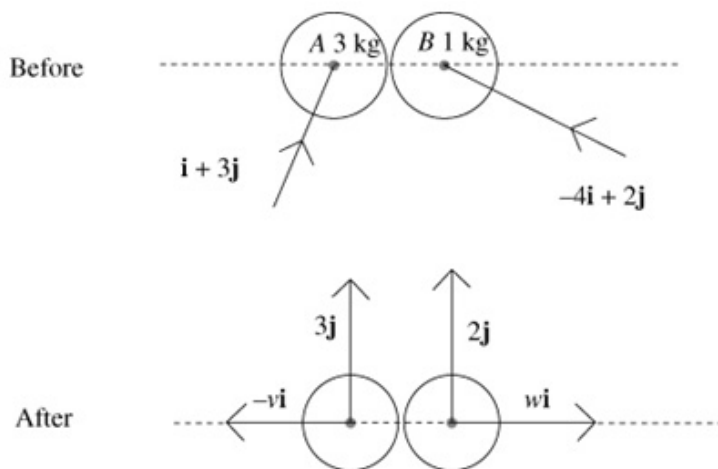
Elastic collisions in two dimensions

Exercise B, Question 10

Question:

A smooth sphere A of mass 3 kg is moving on a smooth horizontal surface with velocity $(\mathbf{i} + 3\mathbf{j})\text{ m s}^{-1}$. Another smooth sphere B of mass 1 kg and the same radius as A is moving on the same surface with velocity $(-4\mathbf{i} + 2\mathbf{j})\text{ m s}^{-1}$. The spheres collide when their line of centres is parallel to \mathbf{i} . The coefficient of restitution between the spheres is $\frac{3}{4}$. Find the speeds of both spheres after the impact.

Solution:



Line of centres parallel to $\mathbf{i} \Rightarrow$ no change in the components of velocity parallel to \mathbf{j}
 conservation of momentum: $3 \times 1 - 1 \times 4 = 1 \times w - 3 \times v = -1$

law of restitution: $v + w = \frac{3}{4}(4 + 1), 4v + 4w = 15$

$$4w - 12v = -4$$

$$16v = 19$$

$$v = \frac{19}{16}, w = \frac{41}{16}$$

After the impact, speed of $A = \sqrt{3^2 + \left(\frac{19}{16}\right)^2} \approx 3.23\text{ m s}^{-1}$,

speed of $B = \sqrt{2^2 + \left(\frac{41}{16}\right)^2} \approx 3.25\text{ m s}^{-1}$

Solutionbank M4

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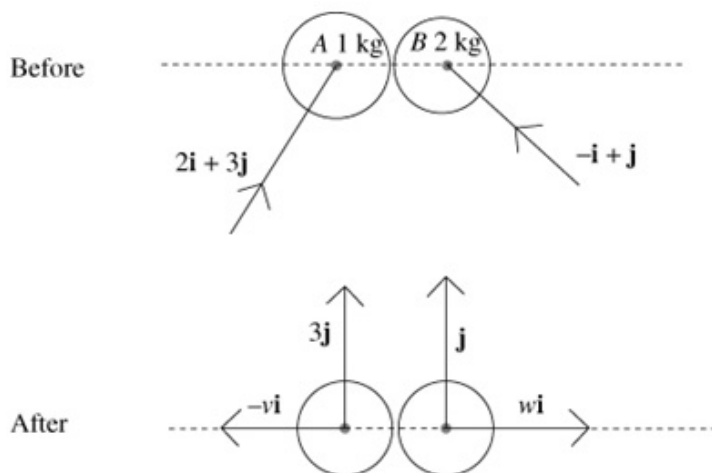
Elastic collisions in two dimensions

Exercise B, Question 11

Question:

A smooth sphere A of mass 1 kg is moving on a smooth horizontal surface with velocity $(2\mathbf{i} + 3\mathbf{j})\text{ m s}^{-1}$. Another smooth sphere B of mass 2 kg and the same radius as A is moving on the same surface with velocity $(-\mathbf{i} + \mathbf{j})\text{ m s}^{-1}$. The spheres collide when their line of centres is parallel to \mathbf{i} . The coefficient of restitution between the spheres is $\frac{3}{5}$. Find the kinetic energy lost in the impact.

Solution:



Line of centres parallel to $\mathbf{i} \Rightarrow$ no change in the components of velocity parallel to \mathbf{j}
 conservation of momentum: $1 \times 2 - 2 \times 1 = 2 \times w - 1 \times v = 0$

$$\text{law of restitution: } v + w = \frac{3}{5}(2 + 1)$$

$$2w - v = 0, 3w = \frac{9}{5}, w = \frac{3}{5}$$

$$v = \frac{6}{5}$$

$$\text{K.E. lost} = \frac{1}{2} \times 1 \times \left(2^2 - \left(\frac{6}{5} \right)^2 \right) + \frac{1}{2} \times 2 \times \left(1^2 - \left(\frac{3}{5} \right)^2 \right) = \frac{48}{25} = 1.92 \text{ J}$$

Components of velocity unchanged parallel to $\mathbf{j} \Rightarrow$ all K.E. lost parallel to \mathbf{i} .

Solutionbank M4

Edexcel AS and A Level Modular Mathematics

Elastic collisions in two dimensions

Exercise B, Question 12

Question:

Two small smooth spheres A and B have equal radii. The mass of A is m kg and the mass of B is $2m$ kg. The spheres are moving on a smooth horizontal plane and they collide. Immediately before the collision the velocity of A is $(2\mathbf{i} + 5\mathbf{j})\text{m s}^{-1}$ and the velocity of B is $(3\mathbf{i} - \mathbf{j})\text{m s}^{-1}$. Immediately after the collision the velocity of A is $(3\mathbf{i} + 2\mathbf{j})\text{m s}^{-1}$. Find

- the velocity of B immediately after the collision,
- a unit vector parallel to the line of centres of the spheres at the instant of the collision.

Solution:

a Conservation of momentum $\Rightarrow m(2\mathbf{i} + 5\mathbf{j}) + 2m(3\mathbf{i} - \mathbf{j}) = m(3\mathbf{i} + 2\mathbf{j}) + 2m\mathbf{v}$

$$2\mathbf{v} = \mathbf{i}(2 + 2 \times 3 - 3) + \mathbf{j}(5 - 2 \times 1 - 2) = 5\mathbf{i} + \mathbf{j}$$

$$\mathbf{v} = \frac{5}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$$

b Impulse on $A = m((3\mathbf{i} + 2\mathbf{j}) - (2\mathbf{i} + 5\mathbf{j})) = m(\mathbf{i} - 3\mathbf{j})$

$$\Rightarrow \text{line of centres parallel to } \frac{1}{\sqrt{10}}(\mathbf{i} - 3\mathbf{j})$$

Solutionbank M4

Edexcel AS and A Level Modular Mathematics

Elastic collisions in two dimensions

Exercise B, Question 13

Question:

Two small smooth spheres A and B have equal radii. The mass of A is $3m$ kg and the mass of B is m kg. The spheres are moving on a smooth horizontal plane and they collide. Immediately before the collision the velocity of A is $(3\mathbf{i} - 5\mathbf{j})\text{ m s}^{-1}$ and the velocity of B is $(4\mathbf{i} + \mathbf{j})\text{ m s}^{-1}$. Immediately after the collision the velocity of A is $(4\mathbf{i} - 4\mathbf{j})\text{ m s}^{-1}$. Find

- the speed of B immediately after the collision,
- the kinetic energy lost in the collision.

Solution:

$$\text{a Conservation of momentum} \Rightarrow 3m(3\mathbf{i} - 5\mathbf{j}) + m(4\mathbf{i} + \mathbf{j}) = 3m(4\mathbf{i} - 4\mathbf{j}) + m\mathbf{v}$$

$$\mathbf{v} = \mathbf{i}(3 \times 3 + 4 - 3 \times 4) + \mathbf{j}(-3 \times 5 + 1 + 3 \times 4) = \mathbf{i} - 2\mathbf{j}$$

$$\text{Speed of } B \text{ is } \sqrt{1^2 + 2^2} = \sqrt{5} \text{ m s}^{-1}$$

$$\text{b K.E. lost} = \frac{3m}{2}((3^2 + 5^2) - (4^2 + 4^2)) + \frac{m}{2}((4^2 + 1^2) - 5)$$

$$= \frac{m}{2}(3(34 - 32) + (17 - 5)) = \frac{m}{2}(6 + 12) = 9m \text{ J}$$

Solutionbank M4

Edexcel AS and A Level Modular Mathematics

Elastic collisions in two dimensions

Exercise B, Question 14

Question:

Two small smooth spheres A and B have equal radii. The mass of A is $2m$ kg and the mass of B is m kg. The spheres are moving on a smooth horizontal plane and they collide. Immediately before the collision the velocity of A is $(2\mathbf{i} + 5\mathbf{j})\text{m s}^{-1}$ and the velocity of B is $(2\mathbf{i} - 2\mathbf{j})\text{m s}^{-1}$. Immediately after the collision the velocity of A is $(3\mathbf{i} + 4\mathbf{j})\text{m s}^{-1}$. Find

- the velocity of B immediately after the collision,
- the coefficient of restitution between the two spheres.

Solution:

a Conservation of momentum $\Rightarrow 2m(2\mathbf{i} + 5\mathbf{j}) + m(2\mathbf{i} - 2\mathbf{j}) = 2m(3\mathbf{i} + 4\mathbf{j}) + m\mathbf{v}$
 $\mathbf{v} = \mathbf{i}(2 \times 2 + 2 - 2 \times 3) + \mathbf{j}(2 \times 5 - 2 - 2 \times 4) = 0$

- b B is brought to a halt in the collision \Rightarrow the line of centres must be parallel to the original direction of motion of B , i.e. $\frac{\sqrt{2}}{2}(\mathbf{i} - \mathbf{j})$

In this direction,

$$\text{speed of } A \text{ before} = ((2\mathbf{i} + 5\mathbf{j}) \cdot \frac{\sqrt{2}}{2}(\mathbf{i} - \mathbf{j})) = \frac{\sqrt{2}}{2}(2 - 5) = -3\frac{\sqrt{2}}{2}$$

$$\text{speed of } A \text{ after} = (3\mathbf{i} + 4\mathbf{j}) \cdot \frac{\sqrt{2}}{2}(\mathbf{i} - \mathbf{j}) = \frac{\sqrt{2}}{2}(3 - 4) = -\frac{\sqrt{2}}{2}$$

$$\text{speed of } B \text{ before} = 2\sqrt{2}$$

$$\text{speed of } B \text{ after} = 0$$

Therefore the impact law gives $\frac{\frac{\sqrt{2}}{2}}{3\frac{\sqrt{2}}{2} + 2\sqrt{2}} = e = \frac{1}{7}$

Solutionbank M4

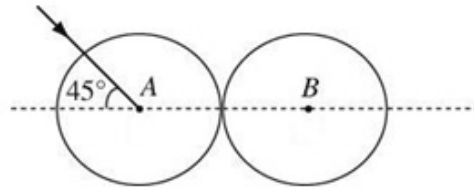
Edexcel AS and A Level Modular Mathematics

Elastic collisions in two dimensions

Exercise B, Question 15

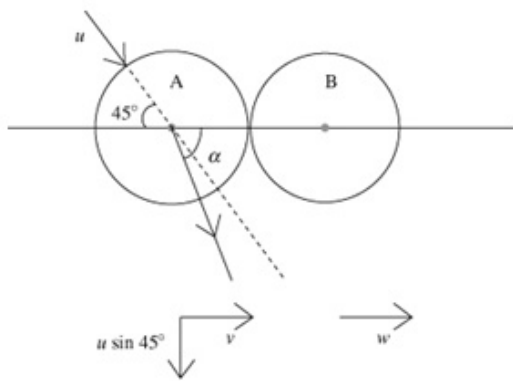
Question:

A smooth uniform sphere A , moving on a smooth horizontal table, collides with an identical sphere B which is at rest on the table. When the spheres collide the line joining their centres makes an angle of 45° with the direction of motion of A , as shown in the diagram. The coefficient of restitution between the spheres is e . The direction of motion of A is deflected through an angle θ by the collision. Show that $\tan \theta = \frac{1+e}{3-e}$.



The direction of motion of A is deflected through an angle θ by the collision. Show that $\tan \theta = \frac{1+e}{3-e}$.

Solution:



Parallel to the line of centres, using conservation of momentum and the law of restitution gives $mu \cos 45^\circ = mv + mw$ and $w - v = eu \cos 45^\circ$

By subtracting

$$2v = u \cos 45^\circ (1 - e)$$

$$v = \frac{u\sqrt{2}(1-e)}{4}$$

$$\Rightarrow \tan \alpha = \frac{u \sin 45^\circ}{\left(\frac{u\sqrt{2}(1-e)}{4}\right)} = \frac{2}{1-e}$$

$$\theta = \alpha - 45^\circ \Rightarrow \tan \theta = \frac{\tan \alpha - \tan 45^\circ}{1 + \tan \alpha \tan 45^\circ} = \frac{\frac{2}{1-e} - 1}{1 + \frac{2}{1-e}} = \frac{2 - 1 + e}{1 - e + 2} = \frac{1 + e}{3 - e}$$

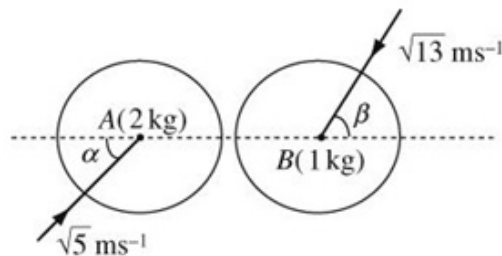
Solutionbank M4

Edexcel AS and A Level Modular Mathematics

Elastic collisions in two dimensions

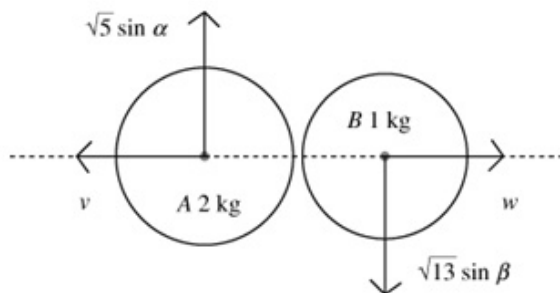
Exercise B, Question 16

Question:



Two smooth uniform spheres A and B of equal radius have masses 2 kg and 1 kg respectively. They are moving on a smooth horizontal plane when they collide. Immediately before the collision the speed of A is $\sqrt{5}\text{ m s}^{-1}$ and the speed of B is $\sqrt{13}\text{ m s}^{-1}$. When they collide the line joining their centres makes an angle α with the direction of motion of A and an angle β with the direction of motion of B , where $\tan \alpha = \frac{1}{2}$ and $\tan \beta = \frac{3}{2}$, as shown in the diagram above. The coefficient of restitution between A and B is $\frac{1}{2}$. Find the speed of each sphere after the collision.

Solution:



Before collision, components of velocity of A are 1 m s^{-1} perpendicular to the lines of centres and 2 m s^{-1} parallel to the line. The components of the velocity of B are 3 m s^{-1} perpendicular to the line, and 2 m s^{-1} parallel to it.

conservation of momentum: $2 \times 2 - 1 \times 2 = 1 \times w - 2 \times v$, $2 = w - 2v$

law of restitution: $w + v = e(2 + 2)$, $w + v = 4e = 2$

Solving the simultaneous equations $\Rightarrow w = 2, v = 0$

\Rightarrow speed of A is 1 m s^{-1} and speed of B is $\sqrt{3^2 + 2^2} = \sqrt{13}\text{ m s}^{-1}$

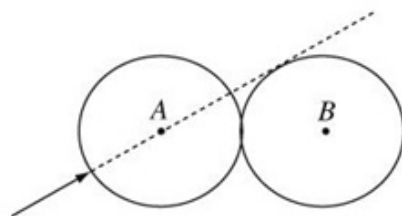
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Elastic collisions in two dimensions

Exercise B, Question 17

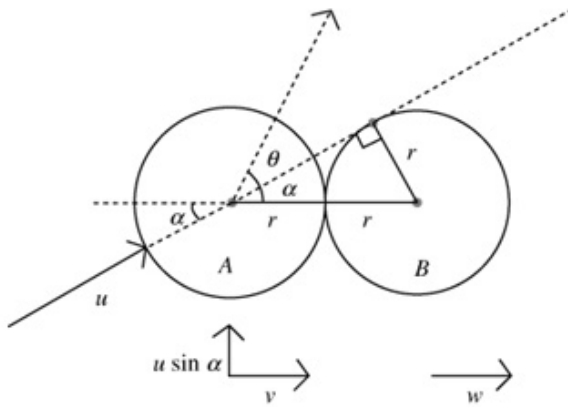
Question:



A smooth uniform sphere B is at rest on a smooth horizontal plane, when it is struck by an identical sphere A moving on the plane. Immediately before the impact, the line of motion of the centre of A is tangential to the sphere B , as shown in the diagram above. The coefficient of restitution between the spheres is $\frac{1}{2}$. The direction of motion of A is turned through an angle θ by the impact.

Show that $\tan \theta = \frac{3\sqrt{3}}{7}$.

Solution:



Tangent perpendicular to radius $\Rightarrow \sin \alpha = \frac{1}{2}$

Initial components of velocity of A are $u \cos \alpha$ parallel to the line of centres, and $u \sin \alpha$ perpendicular to the line of centres.

momentum $\Rightarrow mu \cos \alpha = mv + mw$

$$u \cos \alpha = v + w$$

impact $\Rightarrow w - v = eu \cos \alpha$

Subtracting gives

$$2v = u \cos \alpha - eu \cos \alpha \quad v = \frac{u \cos \alpha \left(1 - \frac{1}{2}\right)}{2} = \frac{u \times \frac{\sqrt{3}}{2} \times \frac{1}{2}}{2} = \frac{u\sqrt{3}}{8}$$

$$\Rightarrow \tan(\theta + \alpha) = \frac{u \sin \alpha}{v} = \frac{\left(\frac{u}{2}\right)}{\left(\frac{u\sqrt{3}}{8}\right)} = \frac{4}{\sqrt{3}}$$

$$\tan \theta = \frac{\tan(\theta + \alpha) - \tan \alpha}{1 + \tan(\theta + \alpha) \tan \alpha} = \frac{\frac{4}{\sqrt{3}} - \frac{1}{\sqrt{3}}}{1 + \frac{4}{\sqrt{3}} \times \frac{1}{\sqrt{3}}} = \frac{\left(\frac{3}{\sqrt{3}}\right)}{\left(\frac{3+4}{3}\right)} = \frac{3\sqrt{3}}{7}$$

Solutionbank M4

Edexcel AS and A Level Modular Mathematics

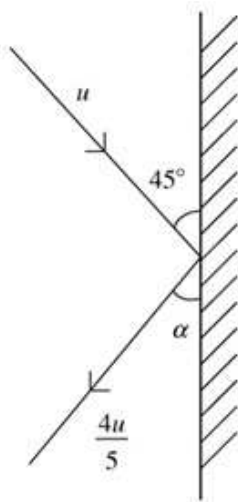
Elastic collisions in two dimensions

Exercise C, Question 1

Question:

A smooth sphere S is moving on a smooth horizontal plane with speed u when it collides with a smooth fixed vertical wall. At the instant of collision the direction of motion of S makes an angle of 45° with the wall. Immediately after the collision the speed of S is $\frac{4}{5}u$. Find the coefficient of restitution between S and the wall.

Solution:



$$R \uparrow: \frac{4u}{5} \cos \alpha = u \cos 45^\circ$$

$$\text{law of restitution } \leftrightarrow: \frac{4u}{5} \sin \alpha = eu \sin 45^\circ$$

squaring and adding:

$$\frac{16u^2}{25} = u^2 \left(\frac{1}{2} + \frac{e^2}{2} \right)$$

$$\frac{32}{25} = 1 + e^2$$

$$\frac{7}{25} = e^2, e = \frac{\sqrt{7}}{5}$$

Solutionbank M4

Edexcel AS and A Level Modular Mathematics

Elastic collisions in two dimensions

Exercise C, Question 2

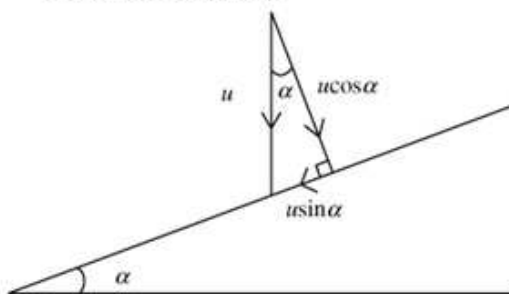
Question:

A small smooth ball of mass $\frac{1}{2}$ kg is falling vertically. The ball strikes a smooth plane which is inclined at an angle α to the horizontal, where $\tan \alpha = \frac{5}{12}$. Immediately before striking the plane the ball has speed 5.2 m s^{-1} . The coefficient of restitution between ball and plane is $\frac{1}{4}$. Find

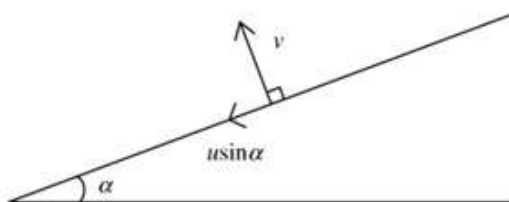
- the speed, to 3 significant figures, of the ball immediately after the impact,
- the magnitude of the impulse received by the ball as it strikes the plane.

Solution:

Before the collision:



After the collision:



- Considering the component of velocity parallel to the plane:

$$u \sin \alpha = 5.2 \times \frac{5}{13} = 2$$

Perpendicular to the plane:

$$v = eu \cos \alpha = \frac{1}{4} \times 5.2 \times \frac{12}{13} = 1.2$$

$$\text{speed} = \sqrt{2^2 + 1.2^2} = \sqrt{5.44} = 2.33 \text{ m s}^{-1}$$

- Impulse = $\frac{1}{2}(1.2 - (-4.8)) = 3 \text{ Ns}$

Solutionbank M4

Edexcel AS and A Level Modular Mathematics

Elastic collisions in two dimensions

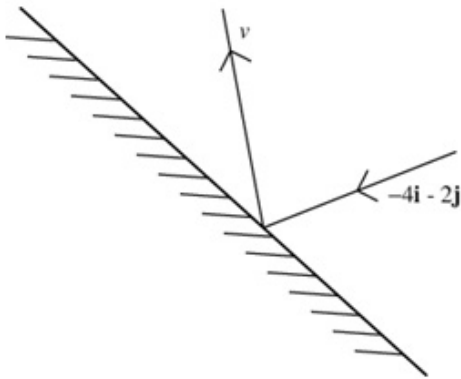
Exercise C, Question 3

Question:

A small smooth ball of mass 500 g is moving in the xy -plane and collides with a smooth fixed vertical wall which contains the line $x + y = 3$. The velocity of the ball just before impact is $(-4\mathbf{i} - 2\mathbf{j})\text{ms}^{-1}$. The coefficient of restitution between the sphere and the wall is $\frac{1}{2}$. Find

- the velocity of the ball immediately after the impact,
- the kinetic energy lost as a result of the impact.

Solution:



- a Suppose that $\mathbf{v} = \mathbf{a} + \mathbf{b}$ where \mathbf{a} is parallel to the wall and \mathbf{b} is perpendicular to the wall.

$\frac{1}{\sqrt{2}}(-\mathbf{i} + \mathbf{j})$ is a unit vector parallel to the

wall and $\frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j})$ is a unit vector perpendicular to the wall.

$$\begin{aligned} \nwarrow \mathbf{a} &= \left[(-4\mathbf{i} - 2\mathbf{j}) \cdot \frac{1}{\sqrt{2}}(-\mathbf{i} + \mathbf{j}) \right] \frac{1}{\sqrt{2}}(-\mathbf{i} + \mathbf{j}) \\ &= \frac{1}{\sqrt{2}}(4 - 2) \times \frac{1}{\sqrt{2}}(-\mathbf{i} + \mathbf{j}) = (-\mathbf{i} + \mathbf{j}) \end{aligned}$$

$$\begin{aligned} \nearrow \mathbf{b} &= -\frac{1}{2} \left[(-4\mathbf{i} - 2\mathbf{j}) \cdot \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) \right] \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) \\ &= -\frac{1}{2} \times \frac{1}{\sqrt{2}}(-4 - 2) \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) \\ &= \frac{3}{2}(\mathbf{i} + \mathbf{j}) \end{aligned}$$

$$\text{So } \mathbf{v} = (-\mathbf{i} + \mathbf{j}) + \frac{3}{2}(\mathbf{i} + \mathbf{j}) = \frac{1}{2}\mathbf{i} + \frac{5}{2}\mathbf{j}$$

b K.E. before impact = $\frac{1}{2} \times \frac{1}{2} \times (4^2 + 2^2) = 5$

$$\text{K.E. after impact} = \frac{1}{2} \times \frac{1}{2} \times \left(\left(\frac{1}{2} \right)^2 + \left(\frac{5}{2} \right)^2 \right) = \frac{1}{4} \times \frac{26}{4} = \frac{13}{8}$$

$$\text{K.E. lost} = 5 - \frac{13}{8} = 3.375 \text{ J}$$

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Elastic collisions in two dimensions

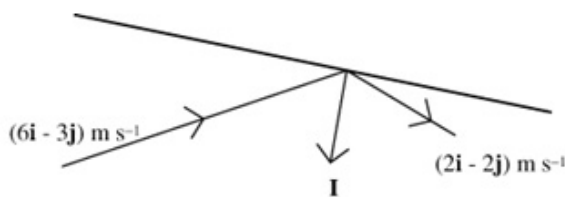
Exercise C, Question 4

Question:

A small smooth sphere of mass m is moving with velocity $(6\mathbf{i} + 3\mathbf{j})\text{ m s}^{-1}$ when it hits a smooth wall. It rebounds from the wall with velocity $(2\mathbf{i} - 2\mathbf{j})\text{ m s}^{-1}$. Find

- the magnitude and direction of the impulse received by the sphere,
- the coefficient of restitution between the sphere and the wall.

Solution:



$$\begin{aligned} \text{a } \mathbf{I} &= m\mathbf{v} - m\mathbf{u} \\ &= m\{(2\mathbf{i} - 2\mathbf{j}) - (6\mathbf{i} + 3\mathbf{j})\} \\ &= m(-4\mathbf{i} - 5\mathbf{j}) \end{aligned}$$

The impulse has magnitude $m\sqrt{16 + 25} = m\sqrt{41}$ Ns in the direction parallel to the unit vector $\frac{1}{\sqrt{41}}(-4\mathbf{i} - 5\mathbf{j})$.

- Component of $(6\mathbf{i} + 3\mathbf{j})$ parallel to the impulse

$$\begin{aligned} &= [(6\mathbf{i} + 3\mathbf{j}) \cdot \frac{1}{\sqrt{41}}(-4\mathbf{i} - 5\mathbf{j})] \times \frac{1}{\sqrt{41}}(-4\mathbf{i} - 5\mathbf{j}) \\ &= \frac{1}{\sqrt{41}}(-24 - 15) \times \frac{1}{\sqrt{41}}(-4\mathbf{i} - 5\mathbf{j}) \end{aligned}$$

Component of $(2\mathbf{i} - 2\mathbf{j})$ parallel to the impulse

$$\begin{aligned} &= [(2\mathbf{i} - 2\mathbf{j}) \cdot \frac{1}{\sqrt{41}}(-4\mathbf{i} - 5\mathbf{j})] \times \frac{1}{\sqrt{41}}(-4\mathbf{i} - 5\mathbf{j}) \\ &= \frac{1}{\sqrt{41}}(-8 + 10) \times \frac{1}{\sqrt{41}}(-4\mathbf{i} - 5\mathbf{j}) \end{aligned}$$

law of restitution

$$\begin{aligned} \frac{2}{\sqrt{41}} &= e \times \frac{39}{\sqrt{41}} \\ e &= \frac{2}{39} \end{aligned}$$

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Edexcel AS and A Level Modular Mathematics

Elastic collisions in two dimensions

Exercise C, Question 5

Question:

Two small smooth spheres A and B have equal radii. The mass of A is $4m$ kg and the mass of B is m kg. The spheres are moving on a smooth horizontal plane and they collide. Immediately before the collision the velocity of A is $(2\mathbf{i} + 3\mathbf{j})\text{m s}^{-1}$ and the velocity of B is $(3\mathbf{i} - \mathbf{j})\text{m s}^{-1}$. Immediately after the collision the velocity of A is $(3\mathbf{i} + 2\mathbf{j})\text{m s}^{-1}$. Find

- the velocity of B immediately after the collision,
- a unit vector parallel to the line of centres of the spheres at the instant of the collision.

Solution:

- a Conservation of momentum \Rightarrow

$$4m(2\mathbf{i} + 3\mathbf{j}) + m(3\mathbf{i} - \mathbf{j}) = 4m(3\mathbf{i} + 2\mathbf{j}) + m\mathbf{v}$$

$$\mathbf{v} = \mathbf{i}(4 \times 2 + 1 \times 3 - 4 \times 3) + \mathbf{j}(4 \times 3 - 1 \times 1 - 4 \times 2) = -\mathbf{i} + 3\mathbf{j}$$

- b Impulse on $A = 4m((3\mathbf{i} + 2\mathbf{j}) - (2\mathbf{i} + 3\mathbf{j})) = 4m(\mathbf{i} - \mathbf{j})$

$$\Rightarrow \frac{\sqrt{2}}{2}(\mathbf{i} - \mathbf{j}) \text{ is a unit vector parallel to the line of centres.}$$

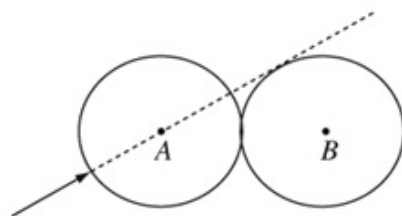
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Elastic collisions in two dimensions

Exercise C, Question 6

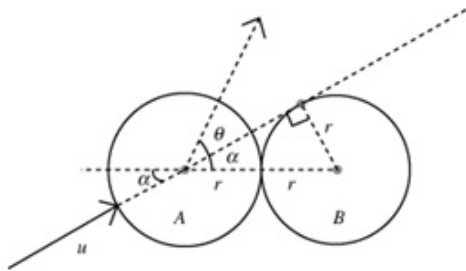
Question:



A smooth uniform sphere B is at rest on a smooth horizontal plane, when it is struck by an identical sphere A moving on the plane. Immediately before the impact, the line of motion of the centre of A is tangential to the sphere B , as shown in the diagram above. The coefficient of restitution between the spheres is $\frac{2}{3}$. The direction of motion of A is turned through an angle θ by the impact.

Show that $\theta = \tan^{-1} \frac{5\sqrt{3}}{9}$.

Solution:



Tangent perpendicular to radius

$$\Rightarrow \sin \alpha = \frac{1}{2}$$

Initial components of velocity of A are $u \cos \alpha$ parallel to the line of centres, and $u \sin \alpha$ perpendicular to the line of centres.

Momentum $\Rightarrow mu \cos \alpha = mv + mw$

$$u \cos \alpha = v + w$$

Impact $\Rightarrow w - v = eu \cos \alpha$

where v is the velocity of A along the line of centres and w the velocity of B along the line of centres immediately after the collision.

Subtracting gives $2v = u \cos \alpha - eu \cos \alpha, v = \frac{u \cos \alpha \left(1 - \frac{2}{3}\right)}{2} = \frac{u \times \frac{\sqrt{3}}{2} \times \frac{1}{3}}{2} = \frac{u\sqrt{3}}{12}$

$$\Rightarrow \tan(\theta + \alpha) = \frac{u \sin \alpha}{v} = \frac{\left(\frac{u}{2}\right)}{\left(\frac{u\sqrt{3}}{12}\right)} = \frac{6}{\sqrt{3}} = 2\sqrt{3}$$

$$\tan \theta = \frac{\tan(\theta + \alpha) - \tan \alpha}{1 + \tan(\theta + \alpha) \tan \alpha} = \frac{2\sqrt{3} - \frac{1}{\sqrt{3}}}{1 + 2\sqrt{3} \times \frac{1}{\sqrt{3}}} = \frac{\left(\frac{6-1}{\sqrt{3}}\right)}{(1+2)} = \frac{5}{3\sqrt{3}} = \frac{5\sqrt{3}}{9}$$

$$\theta = \tan^{-1} \frac{5\sqrt{3}}{9}$$

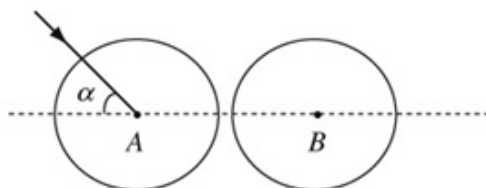
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Elastic collisions in two dimensions

Exercise C, Question 7

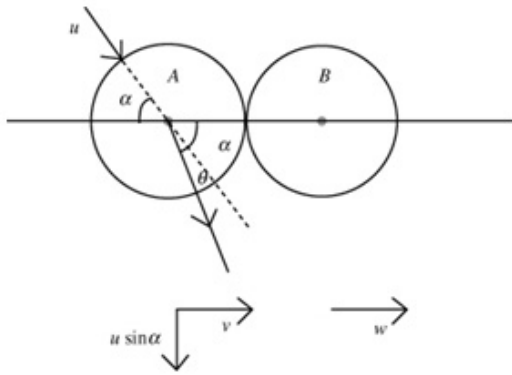
Question:



A smooth uniform sphere A , moving on a smooth horizontal table, collides with a second identical sphere B which is at rest on the table. When the spheres collide the line joining their centres makes an angle of α with the direction of motion of A , as shown in the diagram above. The direction of motion of A is deflected through an angle θ by the collision. Given that $\alpha = \tan^{-1} \frac{3}{4}$ and that the coefficient of restitution between the spheres is e ,

show that $\tan \theta = \frac{6+6e}{17-8e}$.

Solution:



Parallel to the line of centres, using conservation of momentum and the impact law gives

$$mu \cos \alpha = mv + mw$$

$$\text{and } w - v = eu \cos \alpha$$

By subtracting,

$$2v = u \cos \alpha \times (1 - e)$$

$$v = \frac{4u(1-e)}{10} = \frac{2u(1-e)}{5}$$

$$\begin{aligned} \Rightarrow \tan(\theta + \alpha) &= \frac{u \sin \alpha}{\left(\frac{2u(1-e)}{5}\right)} \\ &= \frac{3}{2(1-e)} \end{aligned}$$

$$\tan \theta = \frac{\tan(\theta + \alpha) - \tan \alpha}{1 + \tan(\theta + \alpha) \tan \alpha} = \frac{\frac{3}{2(1-e)} - \frac{3}{4}}{1 + \frac{3}{2(1-e)} \times \frac{3}{4}} = \frac{12 - 6(1-e)}{8(1-e) + 9} = \frac{6 + 6e}{17 - 8e}$$

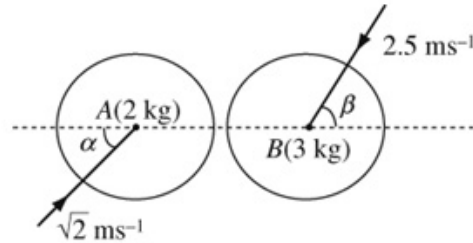
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Elastic collisions in two dimensions

Exercise C, Question 8

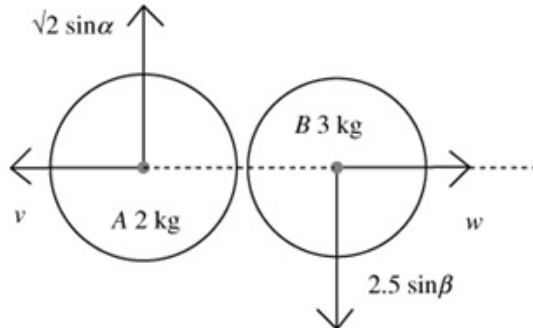
Question:



Two smooth uniform spheres A and B of equal radius have masses 2 kg and 3 kg respectively. They are moving on a smooth horizontal plane when they collide. Immediately before the collision the speed of A is $\sqrt{2}\text{ m s}^{-1}$ and the speed of B is 2.5 m s^{-1} . When they collide the line joining their centres makes an angle α with the direction of motion of A and an angle β with the direction of motion of B , where $\tan \alpha = 1$ and $\tan \beta = \frac{3}{4}$ as shown in the diagram. The coefficient of restitution between A and B is $\frac{2}{3}$.

Find the speed of each sphere after the collision.

Solution:



Before the collision, the components of the velocity of A are 1 m s^{-1} perpendicular to the line of centres and 1 m s^{-1} parallel to the line.

The components of the velocity of B are 1.5 m s^{-1} perpendicular to the line, and 2 m s^{-1} parallel to it.

Conservation of momentum: $2 \times 1 - 3 \times 2 = 3 \times w - 2 \times v$, $-4 = 3w - 2v$

Law of restitution: $w + v = e(1 + 2)$, $w + v = 3e = 2$

Solving the simultaneous equations $-4 = 3w - 2v$ and $4 = 2w + 2v$
 $\Rightarrow w = 0, v = 2$

\Rightarrow speed of A is $\sqrt{1^2 + 2^2} = \sqrt{5}\text{ m s}^{-1}$ and speed of B is 1.5 m s^{-1}

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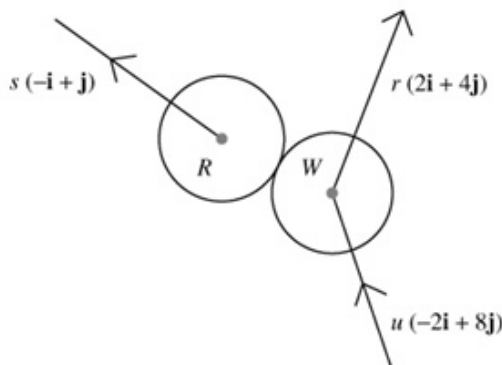
Elastic collisions in two dimensions

Exercise C, Question 9

Question:

A red ball is stationary on a rectangular billiard table $OABC$. It is then struck by a white ball of equal mass and equal radius moving with velocity $u(-2\mathbf{i} + 8\mathbf{j})$ where \mathbf{i} and \mathbf{j} are unit vectors parallel to OA and OC respectively. After the impact the velocity of the red ball is parallel to the vector $(-\mathbf{i} + \mathbf{j})$ and the velocity of the white ball is parallel to the vector $(2\mathbf{i} + 4\mathbf{j})$. Prove that the coefficient of restitution between the two balls is $\frac{3}{5}$.

Solution:



Conservation of momentum:

$$u(-2\mathbf{i} + 8\mathbf{j}) = s(-\mathbf{i} + \mathbf{j}) + r(2\mathbf{i} + 4\mathbf{j})$$

$$\Rightarrow -2u = -s + 2r \text{ and } 8u = s + 4r$$

$$\text{Adding } \Rightarrow 6u = 6r, r = u, s = 4u$$

Line of centres is parallel to $(-\mathbf{i} + \mathbf{j})$ (as this is the direction of the impulse on the red ball).

In the direction of the line of centres

$$\text{component of } (-2\mathbf{i} + 8\mathbf{j}) \text{ is } \frac{(-2\mathbf{i} + 8\mathbf{j}) \cdot (-\mathbf{i} + \mathbf{j})}{|(-\mathbf{i} + \mathbf{j})|} = \frac{10\sqrt{2}}{2} = 5\sqrt{2}$$

$$\text{component of } (2\mathbf{i} + 4\mathbf{j}) \text{ is } \frac{2\sqrt{2}}{2} = \sqrt{2} \text{ and component of } (-\mathbf{i} + \mathbf{j}) \text{ is } \sqrt{2}$$

so using law of restitution:

$$4u\sqrt{2} - u\sqrt{2} = e \times 5u\sqrt{2}, 3\sqrt{2} = 5\sqrt{2}e$$

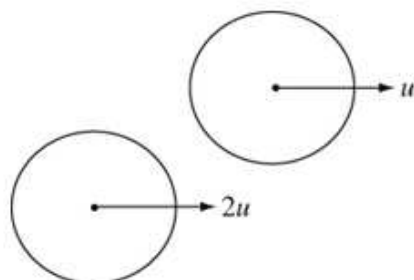
$$e = \frac{3}{5}$$

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Elastic collisions in two dimensions
Exercise C, Question 10

Question:



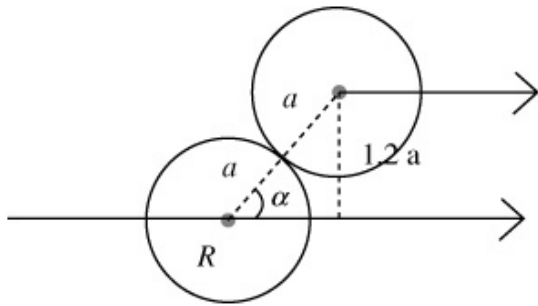
Two uniform spheres, each of mass m and radius a , collide when moving on a horizontal plane. Before the impact the spheres are moving with speeds $2u$ and u , as shown in the diagram.

The centres of the spheres are moving on parallel paths distance $\frac{6a}{5}$ apart.

The coefficient of restitution between the spheres is $\frac{3}{4}$. Find the speeds of the spheres just after the impact, and show that the angle between their paths is then equal to

$$\tan^{-1} \frac{14}{23}.$$

Solution:



Before impact the balls are moving at angle α to the line of centres.

$$\alpha = \sin^{-1} \frac{1.2}{2} = \sin^{-1} \frac{3}{5}$$

momentum:

$$2u \times \frac{4}{5} + u \times \frac{4}{5} = v + w = \frac{12u}{5}$$

law of restitution:

$$w - v = \frac{3}{4} \left(\frac{8u}{5} - \frac{4u}{5} \right) = \frac{3u}{5}$$

Adding:

$$2w = \frac{15u}{5} = 3u, w = \frac{3u}{2}$$

$$\Rightarrow v = \frac{12u}{5} - \frac{3u}{2} = \frac{9u}{10}$$

Speeds are $u \sqrt{\frac{81}{100} + \frac{36}{25}} = u \sqrt{\frac{225}{100}} = \frac{3u}{2}$

and $u \sqrt{\frac{9}{4} + \frac{9}{25}} = u \sqrt{\frac{9 \times 29}{100}} = \frac{3\sqrt{29}}{10} u$

Directions relative to the line of centres are $\tan^{-1} \left(\frac{\frac{6}{5}}{\frac{9}{10}} \right) = \tan^{-1} \frac{4}{3}$ and

$\tan^{-1} \left(\frac{\frac{3}{5}}{\frac{3}{2}} \right) = \tan^{-1} \frac{2}{5}$, so the angle between the paths is

$$\tan^{-1} \left(\frac{\frac{4}{3} - \frac{2}{5}}{1 + \frac{4}{3} \times \frac{2}{5}} \right) = \tan^{-1} \left(\frac{20 - 6}{15 + 8} \right) = \tan^{-1} \frac{14}{23}$$

